Ex 13.1:

1)

$$\lim_{x\to0^+}\frac{\ln(1+2x)}{2x\ln x}=\lim_{x\to0^+}\frac{\ln(1+2x)}{2x}\lim_{x\to0^+}\frac{1}{\ln x}=0$$

2)

$$\left|\frac{\sqrt{x^2+3x}\,\ln(x^2)\sin x}{x\ln x}\right|=\left|2\sqrt{1+\tfrac{3}{x}}\,\sin x\right|\le2\sqrt{1+\tfrac{3}{x}}\le4,\quad\forall x\ge1$$

13.2

Since $e^{x^2}=\int_0^x 2te^{t^2}dt$, then

$$\frac{\int_0^x 2te^{t^2}dt}{\int_0^x e^{t^2}dt}$$

$$=\frac{\int_0^{\sqrt{\ln x}}e^{t^2}dt+\int_{\sqrt{\ln x}}^x 2te^{t^2}dt}{\int_0^{\sqrt{\ln x}}e^{t^2}dt+\int_{\sqrt{\ln x}}^x e^{t^2}dt}$$

$$\geq\frac{2\sqrt{\ln x}\int_{\sqrt{\ln x}}^x e^{t^2}dt}{x\cdot\sqrt{\ln x}+\int_{\sqrt{\ln x}}^x e^{t^2}dt}$$

$$=\frac{2\sqrt{\ln x}}{\frac{x\cdot\sqrt{\ln x}}{\int_{\sqrt{\ln x}}^x e^{t^2}dt}+1}$$

By $\lim_{x\to+\infty}\frac{x\cdot\sqrt{\ln x}}{\int_{\sqrt{\ln x}}^x e^{t^2}dt}=0$, we get

$$\lim_{x\to+\infty}\frac{\int_0^x 2te^{t^2}dt}{\int_0^x e^{t^2}dt}\geq\lim_{x\to+\infty}\frac{2\sqrt{\ln x}}{1+1}=+\infty$$

13.3

1).

$$\lim_{x\to0}x(3+x)\frac{\sqrt{x+3}}{\sqrt{x}\sin\sqrt{x}}\overset{\sin\sqrt{x}\sim\sqrt{x}}{=}\lim_{x\to0}x(3+x)\frac{\sqrt{x+3}}{x}=3\sqrt{3}$$

2).

$e^x-1 \sim x,\ 1-\cos x \sim \frac{x^2}{2}$

$$\lim_{x\to 0}\frac{(1-e^x)(1-\cos x)}{3x^3+2x^4} = \lim_{x\to 0}\frac{-x\cdot\frac{x^2}{2}}{3x^3} = -\frac{1}{6}$$

3).

$$\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}\ln(1+\sin x)}$$

$$= e^{\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\ln(1+\sin x)}{\sin x}}$$

$$= e^1 = e$$

13.4

1).

Since $x-\frac{1}{2}x^2 \leq \ln(1+x) \leq x,\ \forall x \geq 0$, then

$$e^{-n} \geq (\ln(1+e^{-n^2}))^{\frac{1}{n}} \geq (e^{-n^2}-\tfrac{1}{2}e^{-2n^2})^{\frac{1}{n}}$$

$$= e^{-n}(1-\tfrac{1}{2}e^{-n^2})^{\frac{1}{n}},$$

By $\lim_{n\to\infty}(1-\frac{1}{2}e^{-n^2})^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(1-\frac{1}{2e^{n^2}}\right)^{-2e^{n^2}}\right)^{-\frac{1}{n\cdot 2e^{n^2}}} = 1$, hence

$$(\ln(1+e^{-n^2}))^{\frac{1}{n}} \sim e^{-n} \quad \text{as } n\to+\infty$$

2).

$$\left(\frac{e^n}{1+e^{-n}}\right)^n = \frac{e^{n^2}}{(1+e^{-n})^n} = \frac{e^{n^2}}{\left((1+\frac{1}{e^n})^{e^n}\right)^{\frac{n}{e^n}}} \sim e^{n^2}$$

13.5

1). $$\lim_{x\to 0}\left(\frac{x}{\sin x}\right)^{\frac{\sin x}{x-\sin x}} = \lim_{x\to 0}\left(1+\frac{x-\sin x}{\sin x}\right)^{\frac{\sin x}{x-\sin x}} = e$$

2).

$$\lim_{x\to 0}(1+3\tan^2 x)^{\frac{1}{x\sin x}} = \lim_{x\to 0}(1+3\tan^2 x)^{\frac{1}{3\tan^2 x}\cdot\frac{3\tan^2 x}{x\sin x}} = e^3$$

13.6

1).

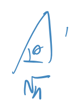
$$\forall x\in[0,\tfrac{\pi}{2}),\ f_n'(x) = -n^2\cos^{n-1}x\sin^2 x + n\cdot\cos^{n+1}x = n^2\cos^{n+1}x\left(\tfrac{1}{n}-\tan^2 x\right)$$

$$= n^2\cos^{n+1}x\left(\tfrac{1}{\sqrt{n}}-\tan x\right)\left(\tfrac{1}{\sqrt{n}}+\tan x\right)$$

Since $\tan x$ is increasing on $[0,\frac{\pi}{2})$, hence $\exists!\ x_n=\arctan\frac{1}{\sqrt{n}}\in(0,\frac{\pi}{2})$ s.t.

$$\tan x_n = \frac{1}{\sqrt{n}}.$$

Then $\forall x\in(0,x_n)$, $f_n'(x)>0$; $\forall x\in(x_n,\frac{\pi}{2})$, $f_n'(x)<0$, thus $x_n=\arctan\frac{1}{\sqrt{n}}$ is the unique point where $f_n(x)$ achieves the maximum.



2).

① $x_n=\arctan\frac{1}{\sqrt{n}}\sim\frac{1}{\sqrt{n}}$

② Since $\tan x_n=\frac{1}{\sqrt{n}}$, then $\sin x_n=\frac{1}{\sqrt{n+1}}$, $\cos x_n=\frac{\sqrt{n}}{\sqrt{n+1}}$.

$$y_n = n\cos^n x_n\sin x_n = n\cdot\left(\frac{n}{n+1}\right)^{\frac{n}{2}}\cdot\frac{1}{\sqrt{n+1}}$$

$$= n\cdot\left(1-\frac{1}{n+1}\right)^{-(n+1)\cdot\frac{n}{-2(n+1)}}\cdot\frac{1}{\sqrt{n+1}}\sim\sqrt{\frac{n}{e}}$$

13.7

$$\left(1+\frac{1}{n}\right)(n+1)^{\frac{1}{n}}-\left(1-\frac{1}{n}\right)(n-1)^{-\frac{1}{n}}$$

$$=\left(\left(1+\frac{1}{n}\right)-\left(1-\frac{1}{n}\right)\right)(n+1)^{\frac{1}{n}}+\left(1-\frac{1}{n}\right)\left((n+1)^{\frac{1}{n}}-(n-1)^{-\frac{1}{n}}\right)$$

$$=\frac{2}{n}(n+1)^{\frac{1}{n}}+\left(1-\frac{1}{n}\right)(n-1)^{-\frac{1}{n}}\left(e^{\frac{1}{n}\ln(n^2-1)}-1\right)$$

where $\frac{2}{n}\cdot(n+1)^{\frac{1}{n}}=\frac{2}{n}\left(1+\frac{\ln(n+1)}{n}+o\left(\frac{\ln(n+1)}{n}\right)\right)=\frac{2}{n}+o\left(\frac{2}{n}\right)$

$$\left(1-\frac{1}{n}\right)(n-1)^{-\frac{1}{n}}\left(e^{\frac{1}{n}\ln(n^2-1)}-1\right)=\left(1-\frac{1}{n}\right)\left(1-\frac{\ln(n-1)}{n}+o\left(\frac{\ln(n-1)}{n}\right)\right)\left(\frac{\ln(n^2-1)}{n}+o\left(\frac{\ln(n^2-1)}{n}\right)\right)=\frac{\ln(n^2-1)}{n}+o\left(\frac{\ln(n^2-1)}{n}\right)$$

$$\Rightarrow \left(1+\frac{1}{n}\right)(n+1)^{\frac{1}{n}} - \left(1-\frac{1}{n}\right)(n-1)^{-\frac{1}{n}}$$

$$= \frac{\ln(n^2-1)}{n} + o\left(\frac{\ln(n^2-1)}{n}\right) \sim \frac{2\ln n}{n} \qquad \square.$$